A COMBINATORIAL DECOMPOSITION OF SIMPLICIAL COMPLEXES

BY

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ABSTRACT

We find a decomposition of simplicial complexes that implies and sharpens the characterization (due to Björner and Kalai) of the f-vector and Betti numbers of a simplicial complex. It generalizes a result of Stanley, who proved the acyclic case, and settles a conjecture of Stanley and Kalai.

1. Introduction

Let Δ be a finite (abstract) simplicial complex on vertex set $V = \{x_1, \ldots, x_n\}$ (i.e. , Δ is a collection of subsets of V such that: $V \subseteq \Delta$; and, if $F \subseteq G$ and $G \in \Delta$, then $F \in \Delta$). Let the **dimension** of $F \in \Delta$ be $\dim F = |F| - 1$, and the **dimension** of Δ be $\dim \Delta = \max\{\dim F\colon F \in \Delta\}$. Also let $d = 1 + \dim \Delta$, so the largest face of Δ has d vertices. Let $f_i = f_i(\Delta) = \#\{F \in \Delta\colon \dim F = i\}$. In particular, $f_{-1} = 1$ for the empty set (unless $\Delta = \emptyset$), f_0 counts the vertices of Δ, and $f_i = 0$ for $i \geq d$. The **f-vector** of Δ is $f(\Delta) = (f_0, \ldots, f_{d-1})$. The same notion of $f_i(\Delta)$ and the f-vector will apply in this paper to every finite collection of sets.

For a simplicial complex Δ, $\tilde{\beta}_i(\Delta) = \dim_K \tilde{H}^i(\Delta; K)$ will denote the ***i*th (reduced) Betti number** of Δ with respect to a fixed field of coefficients K,

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where $\tilde{H}^i(\Delta; K)$ is the ith reduced cohomology group of Δ with respect to K. The **Betti sequence** of Δ is $\tilde{\beta}(\Delta) = (\tilde{\beta}_0, \ldots, \tilde{\beta}_{d-1})$.

Our main result is the following combinatorial decomposition theorem for simplicial complexes.

THEOREM 1.1: *Any (finite) simplicial complex Δ can be written as a disjoint union $\Delta = \Delta' \dot{\cup} B \dot{\cup} \Omega$, where:*

(a) *Δ' is a subcomplex of Δ;*
(b) *$f_i(B) = \tilde{\beta}_i(\Delta)$ and B is an antichain;*
(c) *$\Delta' \dot{\cup} B$ is a subcomplex of Δ; and*
(d) *there exists a bijection η: $\Delta' \to \Omega$ such that for all $F \in \Delta'$ we have $F \subset \eta(F)$ and $|\eta(F) - F| = 1$.*

Theorem 1.1 implies the complete characterization of f-vectors for simplicial complexes with prescribed Betti numbers, which was proved by Björner and Kalai [BK1]. It sharpens and generalizes results by Stanley [St3, Theorem 1.2 and Proposition 2.1] (who proved this result for acyclic simplicial complexes and a weaker result for general simplicial complexes), and settles a conjecture made by Kalai and Stanley [St3, Conjecture 2.2].

Combinatorial decomposition theorems which sharpen extremal combinatorial results are of great interest in combinatorics. A famous example is the decomposition of the Boolean lattice to symmetric antichains. This decomposition implies Sperner's theorem on the size of the largest antichain [GK1]. Another example is an important conjecture made (separately) by Garsia [Ga, Remark 5.2] and Stanley [St2, p. 149], which asserts that every d-dimensional Cohen-Macaulay simplicial complex Δ can be written as a disjoint union of intervals $[S, T]$ so that $\dim T = d$. This conjecture would sharpen Stanley's result that the h-vector of Δ is nonnegative [St1, Corollary 4.3].

In Section 2 we will describe the Kruskal–Katona theorem, which gives a description of f-vectors of simplicial complexes, the Björner–Kalai theorem, which gives a similar description for the case where the Betti numbers are prescribed, and we will show how Theorem 1.1 implies the Björner–Kalai theorem. The proof of Theorem 1.1 is given in Section 3.

2. f-vectors

The f-vector has been characterized for many subclasses of simplicial complexes, and for many generalizations of simplicial complexes. See [Bj] for a survey and extensive bibliography.

For simplicial complexes, the characterization is given by the Kruskal–Katona theorem [Kr, Kat], using the following peculiar function: Given an integer $k \geq 1$, any integer $n \geq 1$ can be written uniquely in the form

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}$$

such that $a_k > \cdots > a_i \geq i > 0$. Define

$$\partial_{k-1}(n) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_i}{i-1}.$$

The function ∂_k has the following combinatorial interpretation (see [GK2, Section 8] or [BK1, Section 2]): Define the **antilexicographic ordering** $\leq_{AL}$ on k-subsets (subsets whose cardinality is k) of V as follows. Arbitrarily order the vertices of V as $x_1 < \cdots < x_n$. Say $S = \{x_{i_1} < \cdots < x_{i_k}\}$ and $T = \{x_{j_1} < \cdots < x_{j_k}\}$ are two k-subsets; then $S <_{AL} T$ if, for some q, we have $i_q < j_q$ and $i_p = j_p$ for $p > q$. A collection C of k-subsets of V is **compressed** if $S \leq_{AL} T$ and $T \in C$ imply $S \in C$. Since $\leq_{AL}$ is a total ordering, there is only one compressed collection of k-subsets of size n; call it I_k^n. The **shadow** of any collection C of k-subsets is

$$\partial C = \{S\colon |S| = k-1,\ S \subset T \text{ for some } T \in C\}.$$

Then ∂I_k^n is also compressed, and $|\partial I_k^n| = \partial_{k-1}(n)$.

THEOREM 2.1 (Kruskal–Katona [Kr, Kat]): *For $f = (f_0, \ldots, f_{d-1})$, the following are equivalent:*

(a) *f is the f-vector of a simplicial complex; and*

(b) *$\partial_k(f_k) \leq f_{k-1}$, for all $k \geq 1$.*

Proof: The simplest proofs are due to Daykin [Da] and Frankl [Fr]. ∎

For further uses and generalizations of the Kruskal–Katona theorem and compression, see [GK2, Section 8].

Björner and Kalai improved upon the Kruskal–Katona theorem by characterizing the f-vector of a simplicial complex with prescribed Betti numbers. The characterization utilizes near-cones.

Definition: A **near-cone** is a simplicial complex Δ containing a vertex v_0 (called the **apex**) with the following property: For any face $F \in \Delta$, if $v_0 \notin F$ and $w \in F$ then

$$(F - \{w\}) \cup \{v_0\} \in \Delta.$$

For any near-cone Δ with apex v_0, let

$$B(\Delta) = \{F \in \Delta\colon F \cup \{v_0\} \notin \Delta\},$$

and let

$$\Delta' = \{F \in \Delta\colon v_0 \notin F, F \cup \{v_0\} \in \Delta\};$$

then

$$\Delta = (v_0 * \Delta') \dot\cup B(\Delta),$$

where $*$ denotes topological join (so $v_0 * \Delta' = \Delta' \dot\cup \{\{v_0\} \dot\cup F\colon F \in \Delta'\}$). Both Δ' and $\Delta' \dot\cup B(\Delta)$ are subcomplexes of Δ. In the case $B(\Delta) = \emptyset$, Δ is a **cone**. Every $F \in B(\Delta)$ is maximal in Δ, so the collection of subsets in $B(\Delta)$ forms an antichain. Further, $f_i(B(\Delta)) = \tilde{\beta}_i(\Delta)$, which follows by the topological process of contracting all the faces of Δ not in $B(\Delta)$ to the apex, leaving a sphere for every face in $B(\Delta)$.

THEOREM 2.2 (Björner–Kalai [BK1, Theorem 1.1]):
For $f = (f_0, \ldots, f_{d-1}), \tilde{\beta} = (\tilde{\beta}_0, \ldots, \tilde{\beta}_{d-1})$, the following are equivalent:

(a) *there is a simplicial complex Δ such that $f = f(\Delta)$ and $\tilde{\beta} = \tilde{\beta}(\Delta)$ (for an arbitrary field K);*

(b) *f is the f-vector of a simplicial complex having the homotopy type of a wedge of, for each i, $\tilde{\beta}_i$ spheres of dimension i;*

(c) *f is the f-vector of a near-cone Δ such that $f_i(B(\Delta)) = \tilde{\beta}_i$;*

(d) *let $\chi_{k-1} = \sum_{j \geq k} (-1)^{j-k}(f_j - \tilde{\beta}_j)$ for $k \geq 0$; then*

$$\chi_{-1} = 1, \quad \textit{and}$$

$$\partial_k(\chi_k + \tilde{\beta}_k) \leq \chi_{k-1} \quad \textit{for all } k \geq 1. \tag{1}$$

An interpretation of equation (1) is

$$\partial_k(\dim Z_k) \leq \dim B_{k-1},$$

where Z_k is the space of k-dimensional cycles ($\{z\colon \partial z = 0\}$) and B_{k-1} is the space of $(k-1)$-dimensional boundaries ($\{b\colon b = \partial y \text{ for some } y\}$). It is also not hard to see that $\chi_k = f_k(\Delta')$ and $\chi_k + \tilde{\beta}_k = f_k(\Delta' \dot{\cup} B)$, where the near-cone in 2.2(c) is $\Delta = (v_0 * \Delta') \dot{\cup} B$.

Proof: (c)⇒(d) is a straightforward combinatorial argument which utilizes the Kruskal–Katona theorem.

(d)⇒(c) is the direct construction of an appropriate near-cone.

(c)⇒(b) follows by remarks following the definition of near-cone.

(b)⇒(a) is clear.

It only remains to show (a)⇒(c). Apply the *algebraic shifting* operation to Δ to obtain a new simplicial complex, Δ^*. This operation has the remarkable property that $f(\Delta) = f(\Delta^*)$ and $\tilde{\beta}(\Delta) = \tilde{\beta}(\Delta^*)$, even though almost no other structure of Δ is preserved. Furthermore, Δ^* is a near-cone, which proves the claim. There is no elementary way of describing Δ^* in terms of Δ. ∎

Björner and Kalai [BK2] generalized this result to polyhedral complexes with a proof by induction that avoids algebraic shifting altogether.

Theorem 1.1 provides a simpler proof of the key step ((a)⇒(c)) of the Björner–Kalai theorem, by constructing, for any simplicial complex Δ, a near-cone with the same f-vector and Betti numbers as Δ.

Alternate proof of (a) ⇒ (c): Using the decomposition $\Delta = \Delta' \dot{\cup} B \dot{\cup} \Omega$ of Theorem 1.1, let $\Delta^* = (v_0 * \Delta') \dot{\cup} B$. It follows immediately that Δ^* is a near-cone, since Δ' and $\Delta' \dot{\cup} B$ are subcomplexes and B is an antichain. Also, Δ^* has the same Betti numbers as Δ, since $\tilde{\beta}_i(\Delta^*) = f_i(B) = \tilde{\beta}_i(\Delta)$. Finally, Δ^* has the same f-vector as Δ because every face $\eta(F) \in \Omega$ is just replaced by $(v_0 * F) \in (v_0 * \Delta') - \Delta'$, which doesn't change the f-vector, since $|\eta(F) - F| = 1$.

Note that *any* near-cone $(v_0 * \Delta') \dot{\cup} B$ trivially satisfies Theorem 1.1 with $\eta(F) = \{v_0\} \dot{\cup} F$.

3. Proof of the main theorem

Let Γ be a directed graph with vertex set V. A **matching** of $A, B \subseteq V$ in Γ is a collection of edges such that each edge is directed from a vertex in A to a vertex in B, and such that every vertex is incident to exactly one of the edges.

The acyclic version ($\operatorname{im} \phi = \ker \phi$) of the following result is [St3, Lemma 1.1].

LEMMA 3.1: *Let Γ be a directed graph on the n-element vertex set X, and let KX be the K-vector space with basis X. Suppose there is a linear transformation ϕ: $KX \to KX$ satisfying*

(a) *if* $x \in X$, *then*

$$\phi(x) \in \operatorname{span}_K\{y \in X\colon (x,y) \text{ is an edge of } \Gamma\};$$

and

(b) $\operatorname{im}\phi \subseteq \ker\phi$ *(i.e.* , $\phi^2 = 0$*).*

Also assume that Y *is a subset of* X *whose image in* $KX/(\operatorname{im}\phi)$ *is a basis for* $KX/(\operatorname{im}\phi)$ *and that* Z *is a subset of* Y *whose image in* $KX/(\ker\phi)$ *is a basis for* $KX/(\ker\phi)$. *Then there is a matching of* Z *and* $X - Y$ *in* Γ.

Proof: Since for any $\phi\colon KX \to KX$ we have $\dim(\ker\phi) + \dim(\operatorname{im}\phi) = n$, it follows $|Z| = n - \dim(\ker\phi) = \dim(\operatorname{im}\phi) = |X - Y|$. By the Marriage Theorem (e.g., [Ry, Ch.5, Thm. 1.1]), it suffices to show that for any $S \subseteq Z$, say with $|S| = k$, there are (at least) k vertices $y_1, \ldots, y_k \in X - Y$ such that for each $1 \le i \le k$ there is an $x \in S$ with (x, y_i) an edge of Γ. Suppose not. Let $S = \{x_1, \ldots, x_k\}$. Then $\phi(x_1), \ldots, \phi(x_k)$ are linearly dependent in KX/KY, since they are all in the span of fewer than k vertices of $X - Y$. Thus some linear combination $x = a_1\phi(x_1) + \cdots + a_k\phi(x_k)$ is in KY ($a_i \in K$, not all $a_i = 0$). Since Y is a basis for $KX/(\operatorname{im}\phi)$ and $x = \phi(a_1x_1 + \cdots + a_kx_k) \in \operatorname{im}\phi$, it follows that x must be 0, so $a_1x_1 + \cdots + a_kx_k \in \ker\phi$. But $x_1, \ldots, x_k$ are linearly independent modulo $\ker\phi$, and hence all $a_i = 0$, a contradiction. ∎

Definition: Let $\Lambda(KV)$ denote the exterior algebra of the vector space KV; it has a K-vector space basis consisting of all the monomials $x^F := x_{i_1} \wedge \cdots \wedge x_{i_k}$ where $F = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq V$. Let I_Δ be the ideal of $\Lambda(KV)$ generated by all $\{x^F\colon F \notin \Delta\}$. The quotient algebra $\Lambda[\Delta] := \Lambda(KV)/I_\Delta$ is called the **exterior face ring** of Δ (over K). A K-vector space basis of $\Lambda[\Delta]$ is the set of all **face monomials** x^F where $F \in \Delta$; it follows that $f_i(\Delta) = \dim_K(\Lambda[\Delta]_i)$. The coboundary operator $\delta\colon \Lambda[\Delta] \to \Lambda[\Delta]$ is then simply right multiplication by $v = x_1 + \cdots + x_n \in \Lambda[\Delta]$, i.e. , $\delta y = y \wedge v$.

Let $\langle \cdot, \cdot \rangle$ be the inner product on $\Lambda(KV)$ such that the set of face monomials forms an orthonormal basis of $\Lambda[\Delta]$. Define the **left interior product** $\llcorner\colon \Lambda^d V \times \Lambda^{k+d} V \to \Lambda^k V$ by

$$\langle u, g \llcorner f \rangle = \langle u \wedge g, f \rangle \quad \forall u \in \Lambda V;$$

then

$$x^F \llcorner x^G = \begin{cases} \pm x^{G-F} & \text{if } F \subseteq G \\ 0 & \text{otherwise,} \end{cases}$$

and the boundary operator $\partial\colon \Lambda[\Delta] \to \Lambda[\Delta]$ is L-multiplication on the left by v, i.e. , $\partial y = v \llcorner y$. This last observation also follows from ∂ and δ being adjoint with respect to $\langle \cdot, \cdot \rangle$, i.e. ,

$$\langle u, \partial f \rangle = \langle \delta u, f \rangle$$

for any $u \in \Lambda[\Delta]_i$, $f \in \Lambda[\Delta]_{i+1}$.

Homology and cohomology are thereby reduced to exterior algebra; this idea was introduced by Kalai [Kal].

LEMMA 3.2: *Let Δ be any simplicial complex with simplicial coboundary operator $\delta\colon \Lambda[\Delta] \to \Lambda[\Delta]$. If $k \in \ker \delta$ and x_j is a vertex of Δ, then $k \wedge x_j \in \operatorname{im} \delta$.*

Proof: Expand k as

$$k = \sum_{F \in \Delta} c_F x^F$$

with $c_F \in K$ (some of the c_F may be 0), and then let

$$k_1 = \sum_{F\colon x_j \in F} c_F x^F$$

and

$$k_2 = \sum_{F\colon x_j \notin F} c_F x^F.$$

Then $k = k_1 + k_2$ and $k_1 \wedge x_j = 0$, so

$$k \wedge x_j = k_2 \wedge x_j.$$

Also, since $k \in \ker \delta$, we have $0 = \delta k = k \wedge v = k_1 \wedge v + k_2 \wedge v$, so

$$k_1 \wedge v = -k_2 \wedge v.$$

We will show that $k \wedge x_j = -k_1 \wedge v \in \operatorname{im} \delta$ by showing that

$$\langle (k \wedge x_j) + (k_1 \wedge v), x^F \rangle = 0 \quad \text{for any } F \in \Delta. \tag{2}$$

Since both x_j and k_1 are multiples of x_j, equation (2) is clear if $x_j \notin F$. Other wise,

$$\begin{aligned} \langle (k \wedge x_j) + (k_1 \wedge v), x^F \rangle &= \langle (k_2 \wedge x_j) - (k_2 \wedge v), x^F \rangle \\ &= \langle k_2 \wedge (x_j - v), x^F \rangle \\ &= 0 \end{aligned}$$

because x_j divides x^F, but not any term of k_2 nor $x_j - v$. ∎

Björner and Kalai proved a similar result with a much more intuitive, topological proof. Let $\operatorname{lk} F := \{G \in \Delta: G \cap F = \emptyset, G \cup F \in \Delta\}$ and $\operatorname{st} F := \{G \in \Delta: G \cup F \in \Delta\}$ denote the **link** and **star** of F, respectively.

REMARK 3.3 (Björner–Kalai [BK1, Remark 4.4]): *For every $x_j \in V$, if $k \in \ker \partial$, then $x_j \llcorner k \in \operatorname{im} \partial$.*

Proof: First, $x_j \llcorner k \in \ker \partial$, and $x_j \llcorner k \in x_j \llcorner \Lambda[\Delta] = \Lambda[\operatorname{lk} x_j] \subseteq \Lambda[\operatorname{st} x_j]$. But $\operatorname{st} x_j$ is acyclic (being a cone with apex x_j) and $(\partial|\operatorname{st} x_j)(y) = \partial y$ for $y \in \operatorname{st} x_j$, so $x_j \llcorner k \in \ker(\partial|\operatorname{st} x_j) = \operatorname{im}(\partial|\operatorname{st} x_j) \subseteq \operatorname{im} \partial$. ∎

This proof does not carry over to Lemma 3.2 since $(\delta|\operatorname{st} x_j)(y) \neq \delta y$ for $y \in \operatorname{lk} x_j \subsetneq \operatorname{st} x_j$. It would be very nice to find a proof of Lemma 3.2 in the spirit of the proof of Remark 3.3.

We are now ready to prove Theorem 1.1. The proof follows the outline of Stanley's proof [St3, Theorem 1.2] of the acyclic case ($\tilde{\beta}_i = 0$).

Proof of Theorem 1.1: Define the **lexicographic ordering** $\leq_L$ on k-subsets of V as follows. Say $S = \{x_{i_1} < \cdots < x_{i_k}\}$ and $T = \{x_{j_1} < \cdots < x_{j_k}\}$ are two k-subsets; then $S <_L T$ if, for some q, we have $i_q < j_q$ and $i_p = j_p$ for $p < q$. Consider the quotient spaces $Q = \Lambda[\Delta]/(\ker \delta)$, and $R = \Lambda[\Delta]/(\operatorname{im} \delta)$. Let L be the lexicographically least basis of Q consisting of face monomials x^F, with respect to the ordering $x_1 \leq \cdots \leq x_n$ of the vertices of Δ, and let M be the lexicographically least set of face monomials such that $L \dot\cup M$ is a basis of R. Thus, if $F \in \Delta$, then $x^F \notin L$ if and only if

$$x^F = a_1 x^{F_1} + \cdots + a_t x^{F_t} + k, \tag{3}$$

where $k \in \ker \delta$, i.e. , $k \wedge v = 0$, and, for each i, we have $a_i \in K$, $F_i \in \Delta$, and $F_i <_L F$. Similarly, if $F \in \Delta$, then $x^F \notin L \dot\cup M$ if and only if

$$x^F = a_1 x^{F_1} + \cdots + a_t x^{F_t} + (y \wedge v), \tag{4}$$

where $y \in \Lambda[\Delta]$ and, for each i, we have $a_i \in K$, $F_i \in \Delta$, and $F_i <_L F$. Finally, let $\Delta' = \{F: x^F \in L\}$ and $B = \{F: x^F \in M\}$.

First we show that Δ' is a subcomplex. Suppose $x^F \notin L$ and $F \subset G$. We need to show that $x^G \notin L$. Multiply equation (3) on the left by x^{G-F}. First note that $x^{G-F} \wedge k \in \ker \delta$ since $\ker \delta$ is an ideal. Further, in $\Lambda[\Delta]$ we have $x^{G-F} \wedge x^{F_i} =$

$\pm x^{(G-F)\dot\cup F_i}$ (or 0 if $(G-F)\cap F_i \neq \emptyset$), and $(G-F)\dot\cup F_i <_L (G-F)\dot\cup F = G$, so equation (3) gives $x^G \notin L$. Thus Δ' is a subcomplex. Similarly, $\Delta'\dot\cup B$ is a subcomplex.

Now let Γ be the directed graph whose vertex set is Δ, and whose edges are the pairs (F, G) with $F \subset G \in \Delta$ and $|G-F| = 1$. Define ϕ: $K\Delta \to K\Delta$ by $\phi = \delta$. By definition of the coboundary operator, ϕ satisfies all the conditions of Lemma 3.1. We can take $Z = \Delta'$ and $Y = \Delta'\dot\cup B$ in Lemma 3.1 by the definitions of Δ' and B, and thus there is a matching η: $\Delta' \to \Delta - (\Delta'\dot\cup B) = \Omega$ satisfying the conditions in (d).

Finally, we prove (b). By construction, $f_i(B) = \tilde\beta_i(\Delta)$. To show that B is an antichain, suppose that $x^F \in M$ and $F \subset G$. We need to show that $x^G \notin M$. Let $x_j \in G - F$, and let $F' = G - \{x_j\}$. Since Δ' is a subcomplex and $F \notin \Delta'$, it follows that $F' \notin \Delta'$ also, so $x^{F'} \notin L$ and

$$x^{F'} = \sum_{i=1}^{t} a_i x^{F_i} + k,$$

where $k \in \ker\delta$, $a_i \in K$, and $F_i <_L F'$.

Thus,

$$\begin{aligned} x^G &= \pm(x^{F'} \wedge x_j) \\ &= \pm\Big(\sum_{i=1}^{t} a_i x^{F_i} \wedge x_j\Big) + (k \wedge x_j) \\ &= \Big(\sum_{\substack{F_i\cup\{x_j\}\in\Delta \\ x_j\notin F_i}} \pm a_i x^{F_i\cup\{x_j\}}\Big) + (k \wedge x_j). \end{aligned}$$

Now

$$F_i \cup \{x_j\} <_L F' \cup \{x_j\} = G,$$

and $k \wedge x_j \in \operatorname{im}\delta$ by Lemma 3.2, so it follows from equation (4) that $x^G \notin M$. ∎

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